

SPHERICAL MEANS OF SOLUTIONS OF
PARTIAL DIFFERENTIAL EQUATIONS IN A CONICAL REGION

by

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ABSTRACT

The spherical means of the solutions of a linear partial differential equation $Lu = f$ in a conical region are studied. The conical region is bounded by a surface generated by curvilinear ξ surfaces. The spherical mean is the average of u over a constant ξ surface. The conditions on the linear differential operator, L , and on the orthogonal coordinates ξ, η, ζ are established so that the spherical mean of the solution subjected to the appropriate boundary and initial conditions can be determined directly as a problem with only one space variable. Conditions are then established so that the spherical mean of the solution in one conical region will be proportional to that of a known solution in another conical region. Applications to various problems of mathematical physics and their physical interpretations are presented.

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1. Introduction

In a study of the solutions of the diffraction of an incident wave by a cone of solid angle Ω with zero normal derivative thereon it was shown [1] that the resultant value at the vertex of the cone is equal to $4\pi/(4\pi-\Omega)$ times the value of the incident wave at the same point in absence of the cone. This relationship was obtained by rederiving Kirchhoff's formula [2] for the wave equation from Green's formula, for the vertex of a cone. Corresponding results can be obtained for the Poisson equation and the diffusion equation. Although the results are different for different differential equations, they can be summarized in a unified statement:

"The value at the vertex of the cone is equal to that of an extended three-dimensional problem without the cone divided by the ratio of the local solid angles of the domains i.e., by the ratio $(4\pi-\Omega)/(4\pi)$. In the extended problem the inhomogeneous term and the initial data are assigned to be zero in the interior of the cone. In case that the normal derivative on the surface of the cone is not zero, the prescribed data should be redistributed and added to the inhomogeneous term for the extended problem."

This unified statement suggests that it may be proved directly from certain properties common to the differential operators of those equations without making use of the Green's theorem and Green's function for each differential equation individually. Such a general approach can be used when the value at the vertex is identified as the limit of the mean value over the local spherical cap exterior to the cone. Therefore, the original plan of presenting the special analyses for each of these differential equations separately is abandoned in favor of the present general approach.

In the next section conditions on the linear differential operator L and the curvilinear orthogonal coordinates ξ, η, ζ are established so that the "spherical" mean of the solution of $Lu = f$ in a domain D subjected to appropriate boundary and initial conditions can be determined directly as the solution of a problem with only one space variable. The domain D is bounded by a "conical" surface S_c generated by the ξ -coordinate lines and by two truncating constant ξ -surfaces S_1 and S_2 . The "spherical" mean $\bar{u}(\xi, t)$ is defined as the mean value of the solution u over a constant ξ -surface in the closed region \bar{D} . Conditions are then established so that the "spherical" mean of the solution in \bar{D} will be proportional to that of a known solution in a different region \bar{D}^* . For the special case that the surface S_1 at $\xi = \xi_1$ degenerates to a point, the vertex of the "cone", the relationship for the spherical means is also valid for the value u at the vertex. These statements are the essence of the Theorems I and II and the two corollaries in Section 2. The aforementioned unified statement on the value at the vertex of a cone is included as a special case of Corollary 2.

The direct determination of the "spherical" mean $\bar{u}(\xi, t)$ provides some important information about the solution and also provides a test of the accuracy of the numerical approximation to u itself (see, e.g. [3], [4].)

Sections 3 to 6 present the application of the theorems to various physical problems. They can be read independently.

Section 3 applies the theorems to biharmonic equations governing the deflection of a circular sector plate subjected to a stationary or moving load. The mean deflections and in particular the deflection at the tip are identified as the symmetric deflections of a circular plate with equivalent loads.

In section 4, the theorems are applied to unsteady three-dimensional diffusion problems in a wedge and in a domain bounded by two cones with a common vertex. The demonstrations of how to obtain results directly from the theorems due to the particular geometries of the domains are, of course, not restricted only to the diffusion problems. The third example shows their applications to the diffusion of pollutants in a moving medium.

In section 5, the theorems are applied to the diffraction of an incident wave by a cone of solid angle Ω . The spherical integral of the resultant wave is equal to that of the incident wave without the cone. The value at the vertex is equal to the value of the incident wave intensified by the factor $4\pi/(4\pi-\Omega)$ as stated in Theorem III. If the incident wave is reflected and diffracted at least by part of the surface of the cone for all time, the primary wave which is known say for $t < 0$ is composed of the incident wave and those reflected and diffracted waves. The spherical integral of the resultant wave for $t > 0$ can again be related directly to that of the primary wave as stated in Theorem IV. Their applications are demonstrated by three examples. They also illustrate how to apply the theorem to bodies which are composed of cones of finite length by making use of the finite speed of propagation.

In section 6, the theorems are applied to the reduced wave equation. In particular, for the diffraction of a plane wave or a cylindrical wave by a cone of infinite length, the Sommerfeld radiation conditions are not applicable. These theorems provide a necessary condition for the resultant wave.

2. Direct Determinability of the Mean-value Over a Curvilinear Coordinate Surface

Let $u(x,y,z,t)$ be the solution of a linear partial differential equation

$$(2.1) \quad Lu = f(x,y,z,t) \quad \text{in } D \quad \text{for } t > 0$$

where x, y , and z are the Cartesian coordinates and t denotes the time variable. Let ξ, η, ζ be a set of curvilinear orthogonal coordinates. The element of arc length is given by

$$(2.2) \quad dx^2 + dy^2 + dz^2 = \alpha^2 d\xi^2 + \beta^2 d\eta^2 + \gamma^2 d\zeta^2$$

The domain D is bounded by two constant ξ -surfaces S_1 and S_2 and a cylindrical surface S_c generated by ξ -coordinate lines with $\xi_2 > \xi > \xi_1$ (see Fig. 1). In the ξ, η, ζ space, the domain D is a right cylinder parallel to the ξ -axis with base area B in the η - ζ plane. The initial and boundary conditions will depend on the operator L . It will be assumed henceforth that these conditions and Eq. (2.1) form a well posed problem. To be more specific, Eq. (2.1) is assumed to be second order in ξ and the standard boundary conditions of the third type [5] will be imposed, i.e.,

$$(2.3) \quad a_j u + (-1)^j b_j u_{\xi} = h_j(\eta, \zeta, t) \quad \text{on } S_j \quad \text{for } j = 1, 2,$$

where a_j and b_j are non-negative functions of t and $a_j + b_j$ is greater than zero. The required boundary condition on S_c will be specified later in Theorem I. The initial conditions at $t = 0$ are

$$(2.4) \quad \frac{\partial^k u}{\partial t^k} = q_k(x, y, z) \quad \text{in } D \quad \text{for } k = 0, 1, \dots, K-1$$

where K is the order of the differential operator L with respect to t . It will be assumed, throughout this paper, that all the given δf , h_j and q_k are piece-wise continuous in their domains of definition, that α , β , γ possess continuous first derivatives and that the solution u has continuous first derivative in the closed region \bar{D} i.e., D and its boundary.

The generalizations of the initial conditions (2.4) and of the boundary conditions (2.3) on S_j when Eq. (2.1) is of any order in ξ will be discussed after Theorem I.

The "spherical" mean $\bar{v}(\xi, t)$ of a function $v(\xi, \eta, \zeta, t)$ is defined as the mean value of v over the constant ξ surface S_ξ in \bar{D} , i.e.,

$$(2.5) \quad \begin{aligned} \bar{v}(\xi, t) &= \iint_{S_\xi} v \, dS / \iint_{S_\xi} dS \\ &= \iint_B v \, \beta \gamma d\eta d\zeta / \iint_B \beta \gamma d\eta d\zeta \end{aligned}$$

Since the "spherical" mean involves only one space variable, it would be of interest to know under what conditions on the operator L and the curvilinear coordinates and with what type of boundary conditions on the surface S_c will it be possible to determine the "spherical" mean $\bar{u}(\xi, t)$ directly prior to the solution $u(\xi, \eta, \zeta, t)$ itself.

The linear differential operator L and the curvilinear coordinates are assumed to fulfill the following four conditions:

$$(2.6) \quad (i) \quad L = L_1 + L_2$$

where L_1 involves only ξ and t .

(ii) $L_2 u$ can be written as the divergence of a vector \underline{A} in a constant ξ surfaces, i.e.,

$$(2.7) \quad L_2 u = \text{div } \underline{A} = \frac{1}{\alpha \beta \gamma} \left[\frac{\partial}{\partial \eta} (\alpha \gamma A_\eta) + \frac{\partial}{\partial \zeta} (\alpha \beta A_\zeta) \right]$$

where A_η and A_ζ are linear combinations of u and its partial derivatives and are identified as the η - and ζ -components of the vector \underline{A} with $A_\xi = 0$.

(iii) The operator L_1 commutes with the averaging operator $(\bar{})$ defined by Eq. (2.5), i.e.,

$$(2.8) \quad \overline{(L_1 u)} = L_1 \bar{u}.$$

When L_1 involves differentiation with respect to ξ , the dependence of the area element of the surface S_ξ on ξ should be separable, i.e.

$$(2.9) \quad \beta \gamma = v(\xi) \mu(\eta, \zeta).$$

(iv) The distance between two constant ξ -surfaces is a constant, i.e.,

$$(2.10) \quad \alpha = \alpha(\xi).$$

On the surface S_c , the required boundary condition is to prescribe the normal component of \underline{A} i.e.

$$(2.11) \quad A_n = g(\xi, \sigma, t) \quad \text{on } S_c$$

where σ is the arc length along the intersection of S_ξ and S_c . If there are more than one boundary condition on S_c in the well-posed problem Eq. (2.11) has to be a linear combination of them.

The divergence theorem is applied to the vector function \underline{A} for the volume ΔD in \bar{D} bounded by two constant ξ surfaces S_ξ and $S_\xi + \Delta\xi$. By utilizing Eq. (2.1), conditions i) and ii) and the boundary condition of Eq. (2.11), the result is

$$\Delta\xi \iint_{S_\xi} \alpha \beta \gamma (L_1 u - f) d\eta d\zeta = \Delta\xi \oint_{\partial S_\xi} \alpha g(\xi, \sigma) d\sigma$$

Due to condition (iv), $\alpha(\xi)$ can be moved outside the integral. After dividing the equation by the volume ΔD , the equation becomes

$$(2.12) \quad \frac{\iint_{S_\xi} (L_1 u) \beta \gamma d\eta d\zeta}{\iint_{S_\xi} \beta \gamma d\eta d\zeta} = \bar{f}(\xi, t) + \tilde{g}(\xi, t)$$

where

$$(2.13) \quad \tilde{g}(\xi, t) = \oint_{\partial S_\xi} g(\xi, \sigma, t) d\sigma / \iint_{S_\xi} \beta \gamma d\eta d\zeta$$

The leftside of Eq. (2.12) is the "spherical" mean of $L_1 u$ which is equal to $L_1 \bar{u}$ on account of condition (iii). Equation (2.12) then becomes the differential equation for \bar{u} . The boundary conditions and the initial conditions

for \bar{u} are supplied by averaging Eqs. (2.3) and (2.4). These results are summarized in the following theorem.

Theorem I

When the differential operator L and the orthogonal curvilinear coordinates fulfill conditions (i) to (iv), the "spherical" mean $\bar{u}(\xi, t)$ of the solution of $Lu = f$ in D subjected to the boundary conditions of Eq. (2.3) and (2.11) and the initial condition of Eq. (2.4) is governed by the following equations:

$$(2.14) \quad \text{D.E.} \quad L_1 u(\xi, t) = \bar{f}(\xi, t) + \tilde{g}(\xi, t), \text{ for } \xi_1 < \xi < \xi_2 \text{ and } t > 0,$$

$$(2.15) \quad \text{B.C.} \quad a_j \bar{u} + b_j \bar{u}_\xi (-1)^j = \bar{h}_j(t), \text{ at } \xi = \xi_j \text{ for } j = 1, 2,$$

$$(2.16) \quad \text{I.C.} \quad \frac{\partial^k \bar{u}}{\partial t^k} = \bar{q}_k(\xi), \text{ at } t = 0 \text{ for } \xi_1 < \xi < \xi_2 \text{ and } k = 0, 1, \dots, K-1.$$

From physical point of view, the operations leading to Eq. (2.14) is equivalent to setting up the "conservation" equation for a thin shell in \bar{D} between surfaces S_ξ and $S_{\xi+\Delta\xi}$ with thickness $\alpha\Delta\xi$. If f is interpreted as the intensity of a spatial source distribution in D and g as a surface distribution of sources over S_c , the inhomogeneous term in Eq. (2.14) for the "spherical" mean represents a spatial source distribution independent of η and ζ with intensity equal to the "spherical" mean of the spatial distribution f plus a redistribution of the source on S_c with constant ξ uniformly over the surface S_ξ .

It should be noted that in Theorem I, the operators on \bar{u} in the initial conditions and in the boundary conditions at $\xi = \xi_1$, and ξ_2 are the same as those for u because these operators are independent of η and ζ . It is evident that if the operators on u in Eq. (2.4) are generalized to be any linear operator independent of η and ζ , then the same operators will appear in the initial conditions on u . Similarly, the boundary conditions (2.3) on S_1 and S_2 can be generalized with total number of conditions equal to the order of Eq. (2.1) in ξ_1 provided that those boundary operators on u are linear and independent of η and ζ . The same operators will again be applied to \bar{u} in its boundary conditions on $\xi = \xi_1$ and ξ_2 in Theorem I.

The geometry of the surface S_c or that of the base area B in the η - ζ plane appears implicitly in the definition of the inhomogeneous terms of Eqs. (2.14) and (2.16) and has no effect on how to construct the solution \bar{u} of those equations. The solution \bar{u} will not be changed if there is an interchange of the surface source distribution g with the spatial distribution f so long as the sum $\bar{f} + \bar{g}$ remains unchanged. In particular, all the surface source g can be replaced by an increment of f in D by the amount \bar{g} with \bar{u} remaining the same. In physical problems, the inhomogeneous term f can contain point sources in D , therefore, the surface source distribution can be absorbed in f directly. In this respect, the discussions can be simplified by imposing a homogeneous boundary condition on S_c , i.e.

$$(2.17) \quad A_n = 0$$

There are cases that $u^*(\xi, \eta, \zeta, t)$ which is the solution of the same set of equations for u but in a different domain D^* , is given or can be obtained

easily. The "spherical" mean $\bar{u}^*(\zeta, t)$ can be computed directly from u^* by the definition of Eq. (2.5). It is therefore of interest to relate the "spherical" mean in D^* to that in D .

Domain D^* is also bounded by two constant ξ surfaces S_1^* and S_2^* with $\xi = \xi_1$ and ξ_2 respectively and by a cylindrical surface generated by ξ -coordinate lines with $\xi_2 > \xi > \xi_1$ with base area B^* in the η - ζ plane.

From condition iii) on the curvilinear coordinates, the ratio of the area of constant- ξ surface in D^* to that in D becomes

$$(2.18) \quad \frac{\iint_{S_\xi} \beta \gamma d\eta d\zeta}{\iint_{S_\xi^*} \beta \gamma d\eta d\zeta} = \frac{\iint_B \mu(\eta, \zeta) d\eta d\zeta}{\iint_{B^*} \mu(\eta, \zeta) d\eta d\zeta} = \Lambda$$

where Λ is a constant.

It will be assumed that the supports in inhomogeneous terms in the differential equation and in the initial conditions lie in $D \cap D^*$, and the supports of the inhomogeneous terms in the boundary conditions on S_j lie in $B \cap B^*$. The solution u^* in the domain D^* is then defined as the associated solution of u in D if they fulfill the same differential equation, the same boundary conditions on S_1 and S_2 , and the same initial conditions, and the homogeneous condition $A_n = 0$ on S_c for u and $A_n^* = 0$ on S_c^* for u^* where \underline{A}^* is the vector \underline{A} with u replaced by u^* . The last condition implies either $g = g^* = 0$ or that the surface source distribution g has been absorbed in f .

Let \bar{v}^* designate the "spherical" mean of v^* over a constant ξ surface in D^* . The mean values of the inhomogeneous terms in the differential equations initial conditions and boundary conditions on S_j^* for \bar{u}^* will be Λ times the values for u , e.g.,

$$(2.19) \quad \bar{f}^* = \Lambda \bar{f}.$$

By theorem I, \bar{u}^* obeys the same equations as \bar{u} with the exception that all the inhomogeneous terms change by a factor Λ , therefore,

$$(2.20) \quad \bar{u}^*(\xi, t) = \Lambda \bar{u}(\xi, t) \quad \text{or} \quad \bar{u}(\xi, t) = \bar{u}^*(\xi, t)/\Lambda.$$

This result is stated as follows:

Theorem II

The "spherical" mean $\bar{u}(\xi, t)$ of the solution u in D is equal to "spherical" mean $\bar{u}^*(\xi, t)$ of the associated solution u^* in D^* divided by the ratio of the area of the constant ξ surface in D to that in D^* .

By using the definition of Eq. (2.5), Eq. (2.20) becomes

$$(2.21) \quad \iint_{S_\xi} u(\xi, \eta, \zeta, t) dS = \iint_{S_\xi^*} u(\xi, \eta, \zeta, t) dS$$

This relationship restated in the following Corollary is useful when one of the areas, say that of S_ξ^* , is not finite.

Corollary I

The integral of $u(\xi, \eta, \zeta, t)$ over a constant ξ -surface in D is equal to that of its associated solution u^* in D^* .

A) Admissible Orthogonal Coordinates

Conditions iii) and iv) i.e. Eqs. (2.9 and 2.10), impose some restrictions on the curvilinear orthogonal coordinates. It is shown in an NYU Report [6] that Eq.(2.10) requires the ξ -coordinate lines to be straight

lines then the unit tangent vector $\underline{\xi}$ is independent of ξ . When ξ is identified as the arc length, $\alpha(\xi)$ equals to 1. It is then shown in [6] that the constant ξ -surfaces and the orthogonal coordinates, which are consistent with Eq. (2.9), have to be one of the following three types:

a) $\underline{\xi}$ is independent of η and ζ . The constant ξ surfaces are parallel planes say normal to x-axis. The variable ξ can be identified as x while η, ζ can be identified as y and z or a pair of orthogonal coordinates in the y-z plane. The domain D is a right cylinder parallel to x-axis (see Fig. 2a).

b) $\underline{\xi}$ is dependent only on η and ζ . The constant ξ surfaces are coaxial circular cylindrical surfaces. If the axis is identified as the z-axis. The orthogonal coordinates can be identified as the cylindrical coordinates (ρ, θ, z) respectively. The domain is bounded by the cylindrical surfaces $\rho = \xi_1$ and $\rho = \xi_2$ and the "conical" surface S_c which can be defined as $S(z, \theta) = 0$, (see Fig. 2b).

c) $\underline{\xi}$ depends on both η and ζ . The variable ξ is identified as the radial distance r to the center and η and ζ can be identified as the spherical angles θ and φ respectively or as two orthogonal coordinates on the spherical surface. The domain D is a cone truncated by spherical surfaces of radii ξ_1 and ξ_2 . (see Fig. 2c).

Under type c, the area S_r of a spherical cap in D is $r^2 \Omega_c$ where Ω_c is the solid angle of the cone. When $\xi_1 = 0$ the spherical cap S_1 at $r = \xi_1$ degenerates to the vertex of the cone and the spherical mean value $\bar{u}(0, t)$ becomes the value at the vertex of the cone. Theorem I will therefore relate the value at the vertex of the cone D to that at the center of a spherically symmetric problem without the conical surface S_c , i.e.,

$$(2.22) \quad u(r = 0, \theta, \varphi, t) = u(0, t)$$

The boundary condition(s) at $r = \xi_1 = 0$ will be the specification of the behavior of u and/or its derivatives with respect to r as required by the order of Eq. (2.1) with respect to ξ . The same condition(s) will be imposed on \bar{u} at $r = 0$.

When D^* is another conical domain of solid angle Ω_c^* the ratio of the area of spherical cap S_r in D to that of S_r^* in D^* is

$$\Lambda = S_r / S_r^* = \Omega_c / \Omega_c^*$$

When the required boundary condition for u on S_c , Eq. (2.11), is homogeneous, Theorem II yields that the spherical mean of the solution $u(r, \theta, \varphi, t)$ in D is equal to that of u^* in D^* divided by Λ which is the ratio of the solid angles, i.e.,

$$(2.23) \quad \bar{u}(r, t) = \bar{u}^*(r, t) / (\Omega_c^* / \Omega_c)$$

At $r = 0$, the mean values \bar{u} , \bar{u}^* are the values of u and u^* respectively at the vertex. The following statement is now evident.

Corollary 2

The solution u at the vertex of a cone D with $0 < r < \xi_2$ are equal to the associated solution u^* at the vertex of cone D^* divided by the ratio of their solid angles.

For the special case that D^* is the whole sphere, $r < \xi_2$ with $\Omega^* = 4\pi$, the associated solution u^* is a solution inside the sphere in absence of a

conical surface. u^* at the center of the sphere is related to the solution u and the vertex of the cone D directly,

$$(2.24) \quad u(r = 0, \theta, \varphi, t) = \frac{4\pi}{\Omega_c} u^* (r = 0, \theta, \varphi, t)$$

This equation is equivalent to the statement in Section 1 which summarizes the results obtained by the applications of Green's formulas and the appropriate functions to several equations of mathematical physics.

It should be pointed out here that conditions for the direct determination of the spherical mean in Theorem I are not necessary. For example, the operator L_2 involving the variables η and ζ can be removed by means of the Stokes Theorem if $L_2 u$ can be expressed as the ξ -component of the curl of a vector. Details of the alternate approach can be found in [6]. They will not be presented in this paper because in most physical problems, the governing equations are usually derived from the conservation of certain quantities and the representation of $L_2 u$ as the divergence of a surface vector is usually expected.

In the next sub-section it will be shown that several equations of mathematical physics fulfill the conditions stated in Theorem I.

B) Equations of Mathematical Physics

For many equations of mathematical physical, namely the Poisson equation, the unsteady diffusion equations, the wave equation, etc., the differential operator involving the space variables is the Laplacian operator, i.e.

$$(2.25) \quad L = \text{div grad} + L_t$$

where L_t involves differentiations with respect to time only.

To compare with the conditions on L , the vector \underline{A} in Eq. (2. 7) can be identified as the projection of $\text{grad } u$ on the η - ζ surface, i.e.

$$(2.26) \quad \underline{A} = \frac{1}{\beta} \frac{\partial u}{\partial \eta} \underline{\eta} + \frac{1}{\gamma} \frac{\partial u}{\partial \zeta} \underline{\zeta}$$

where $\underline{\eta}$ and $\underline{\zeta}$ are unit vectors along the η and ζ -coordinate lines respectively.

From the definition that $Lu = L_1 u + \text{div } \underline{A}$, the operator L_1 is [7]

$$L_1 = \frac{1}{\alpha \beta \gamma} \frac{\partial}{\partial \xi} \left(\frac{\beta \gamma}{\alpha} \frac{\partial}{\partial \xi} \right) + L_t$$

When the coordinates are admissible in the sense defined in the preceding subsection L_1 becomes

$$(2.27) \quad L_1 = \frac{1}{v(\xi)} \frac{\partial}{\partial \xi} \left[v(\xi) \frac{\partial}{\partial \xi} \right] + L_t$$

It is clear that L_1 involves only ξ and t and the operator L defined by (2.25) fulfills the four conditions required for the direct determination of the "spherical" mean. The normal component of \underline{A} on S_c is equal to the normal component of $\text{grad } u$. The required boundary condition on S_c , Eq. (2.11), becomes

$$(2.28) \quad \frac{\partial u}{\partial n} = g(\xi, \sigma, t) \quad \text{on } S_c$$

In the analysis of diffusion through a moving incompressible fluid, the equations for the velocity field \underline{q} is uncoupled from the equation for the diffusion process. For a given velocity field, the operator L is of the form

$$(2.29) \quad L = \partial_t + \underline{q} \cdot \nabla - \Delta$$

Lu can be rewritten as $u_t + \nabla \cdot [\underline{q}u - \nabla u]$. To be consistent with the conditions for Theorem I, it is necessary that the ξ -component of the velocity is independent of η and ζ and that $(\underline{q} \cdot \underline{n})u - (\partial u / \partial n)$ is prescribed on the surface S_c . The second condition is equivalent to the prescription of the flux of u through the surface S_c . For the special case that the surface S_c is a rigid wall, $\underline{q} \cdot \underline{n}$ vanishes on S_c and the required boundary condition on S_c is the prescription of $\partial u / \partial n$ i.e. again Eq. (2.28).

For biharmonic equations, say

$$(2.30) \quad L = \Delta \Delta + L_t$$

the vector \underline{A} in Eq. (2.7) can be identified as

$$(2.31) \quad \underline{A} = \underline{\eta} \left[\frac{1}{\beta} \frac{\partial}{\partial \eta} \Delta u + \beta L_\xi \left(\frac{1}{\beta^2} \frac{\partial u}{\partial \eta} \right) \right] \\ + \underline{\zeta} \left[\frac{1}{\gamma} \frac{\partial}{\partial \zeta} \Delta u + \gamma L_\xi \left(\frac{1}{\gamma^2} \frac{\partial u}{\partial \zeta} \right) \right]$$

where $L_\xi = \frac{1}{v(\xi)} \frac{\partial}{\partial \xi} \left(v \frac{\partial}{\partial \xi} \right)$. From the definition of $Lu = L_1 u + \text{div } \underline{A}$, it can be seen that the operator L_1 is equal to $(L_\xi)^2 + L_t$ and involves only t and ξ . The required boundary condition on S_c in Theorem I becomes

$$(2.32) \quad \underline{A} \cdot \underline{n} = \frac{\partial}{\partial n} \Delta u + \underline{n} \cdot \underline{\eta} \beta L_\xi \left(\frac{1}{\beta^2} \frac{\partial u}{\partial \eta} \right) + \underline{n} \cdot \underline{\zeta} \gamma L_\xi \left(\frac{1}{\gamma^2} \frac{\partial u}{\partial \zeta} \right) = g(\xi, \sigma, t)$$

It is obvious that the operator L in general will not be invariant with respect to a translation of the origin, see for example Eq. (2.29). However, many of the operators do have this invariant property, e.g., Eq. (2.25) and (2.30),

and then Theorem I can be used to yield the mean-value theorem for any interior point and to develop solutions for initial value problems.

C) Connections with the Mean-Value Theorem and Initial Value Problems

In this subsection it is assumed that the operator L fulfills the conditions in Theorem I and possesses the property of invariance with respect to a translation of the origin. Any interior point P_0 of D can be chosen as the origin and Theorem I can be applied to determine directly the spherical mean \bar{u} inside a sphere S_2 lying in D and centered at P_0 . The spherical mean and in particular the value of u at P_0 is related to the boundary data on S_2 by solving a problem with only one space variable. Thus a mean value theorem for Eq. (2.1) is established. Of course the boundary conditions on the conical surface S_c will not appear in this consideration.

For an initial value problem, the domain D is the whole space. Any point P_0 can again be chosen as the origin and Theorem I will relate the solution at P_0 to the initial data by solving directly the equation for the spherical mean. Applications of spherical means to initial value problems in this respect can be found in [5] and [8].

The theorems in this section and the examples in the following sections deal with initial boundary-value problems. Additional examples can be found in [6].

3. The Biharmonic Equation

The biharmonic equation appears frequently in elasticity problems namely in the plane stress or plane strain problems [9] and in the deflection of thin plates [10]. It is also the governing equation for a slow viscous flow [11]. In this section $u(r, \theta, t)$ will represent the deflection of a thin circular sector plate in polar coordinates r, θ . The governing equation is

$$(3.1) \quad \Delta \Delta u + \frac{\partial^2 u}{\partial t^2} = f \quad \text{in } D.$$

The domain D is the circular sector of radius R and angle α , i.e., $0 < r < R$ and $-\alpha/2 < \theta < \alpha/2$. For the two dimensional problem, eq. (2.31) reduces to

$$(3.2) \quad \underline{A} = \underline{\theta} \left\{ \frac{1}{r} \frac{\partial}{\partial \theta} \Delta u + \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial u}{\partial \theta} \right) \right] \right\} \\ = \underline{\theta} \quad A_{\theta}$$

Under theorem I, the two boundary conditions on S_c , i.e., on the two straight edges $\theta = \pm \alpha/2$, should be sufficient to specify A_{θ} . As a mathematical problem, A_{θ} can be specified as one boundary condition and the mean solution will be independent of the other boundary condition.

The two boundary conditions to be imposed for the plate will appear naturally in the formulation of the governing equations from the variational principal [10]. The following terms appear in the line integral along the boundary

$$-M_n \frac{\partial \delta u}{\partial n} + (Q_n - \frac{\partial M_{ns}}{\partial s}) \delta u$$

where M_n , M_{ns} and Q_n are the bending moment, torsional moment and shearing force and n and s are arc lengths normal to and along the boundary respectively. The line integral vanishes when one of the following four pairs

of boundary data are prescribed: (i) u and $\partial u/\partial n$, (ii) u and M_n , (iii) M_n and $(Q_n - \partial M_{ns}/\partial S)$, (iv) $\partial u/\partial n$ and $(Q_n - \partial M_{ns}/\partial S)$. When the boundary conditions are homogeneous, the first three pairs are known as built-in, simple supported and free edge conditions. The fourth pair will be called rotation-constrained edge conditions, e.g., the edge is reinforced by a bar with torsional rigidity much larger than its flexural rigidity to constrain the deflection of the plate.

Along the straight edges of the sector plate, $\theta = \pm \alpha/2$, only the fourth pair of the edge conditions will yield the data on A_θ . They are

$$(3.3) \quad u_\theta = 0 \quad \text{and}$$

$$(3.4) \quad \frac{1}{r} \frac{\partial}{\partial \theta} (\Delta u) + (1 - \nu) \frac{\partial^2}{\partial r^2} \left(\frac{1}{r} \frac{\partial u}{\partial \theta} \right) = 0$$

for $0 < r < R$ where ν denote Poisson ratio in this section. From eq. (3.3), eq. (3.4) reduces to $\partial(\Delta u)/\partial \theta = 0$. From the definition of A_θ in eq. (3.2) it is clear that the combination of eqs. (3.3) and (3.4) yields

$$(3.5) \quad A_\theta = 0$$

on $\theta = \pm \alpha/2$. The two boundary conditions at the tip of the plate, $r = 0$, are

$$(3.6) \quad u \text{ is finite and}$$

$$(3.7) \quad \lim_{r \rightarrow 0} r \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \right] = 0.$$

Equation (3.7) implies there is no concentrated load at the tip, otherwise the limit will be non zero. The boundary conditions along the circular arc $r = R$ can be,

$$(3.8) \quad u = 0 \text{ and } \partial u/\partial r = 0 \text{ for a built-in edge,}$$

(3.9) $u = 0$ and $\partial^2 u / \partial r^2 + (v/r) \partial u / \partial r = 0$ for a simply supported edge for $|\theta| < \alpha/2$. The boundary conditions are at $r = 0$ and $r = R$ do not involve θ , therefore, Theorem I can be applied to the sector plate to determine the mean $\bar{u}(r, t)$ directly and in particular the value at the tip $u(0, \theta, t)$ which is $\bar{u}(0, t)$. The mean \bar{u} in this two dimensional problem is defined as

$$(3.10) \quad \bar{u}_r(r, t) = \int_{-\alpha/2}^{\alpha/2} u(r, \theta, t) d\theta / \alpha$$

The following two examples will deal with the static and unsteady problems respectively.

Example 1. Deflection of a sector plate due to a stationary concentrated load. For a concentrated load P applied at a point (r_0, θ_0) in D the inhomogeneous term f in eq. (3.1) becomes

$$(3.11) \quad f = (P/I) \delta(r-r_0) \delta(\theta - \theta_0) / r$$

where I is the flexural rigidity of the plate. With a built-in edge along the circular arc (eq. 3.8) and rotation-constrained edges along the straight edge (eqs. 3.3, 3.4), the mean deflection $\bar{u}(r)$ is governed by the equation

$$(3.12) \quad \left(r \frac{d}{dr} r \frac{d}{dr} \right)^2 \bar{u} = (P/I) \delta(r-r_0) / (r\alpha)$$

The boundary conditions at $r = 0$ and $r = R$ are given by eqs. (3.6), (3.7) and (3.8) with u replaced \bar{u} . The solution is

$$(3.13) \quad \bar{u}(r) = \frac{P}{4\alpha I} \left\{ (r^2 + r_0^2) \log \frac{r_0}{R} + (R^2 + r^2) \left(1 - \frac{r_0^2}{R^2} \right) / 2 \right. \\ \left. + [r_0^2 - r^2] + (r_0^2 + r^2) \log (r/r_0) \right\} H(r - r_0)$$

where H is the Heaviside unit step function. The deflection at the tip of the sector plate is

$$(3.14) \quad u(0, \theta) = \bar{u}(0) = \frac{P}{4\alpha I} \left[r_0^2 \log \frac{r_0}{R} + (R^2 - r_0^2) / 2 \right]$$

The mean deflection is the same as the symmetric deflection [10] of a circular plate with built-in edge at $r = R$ and a uniformly distributed load along a concentric circle of radius r_0 with total strength $P(2\pi/\alpha)$.

Example 2. Deflection of a sector plate due to a moving load

For a concentrated moving load of strength P , the inhomogeneous term $f(r, t)$ is again given by eq. (3.11), however, $r_0(t)$ and $\theta_0(t)$ are now given functions of t describing the motion of the load. If the boundary conditions on the sector plate are the same as those in example 1, the mean deflection $\bar{u}(r, t)$ is governed by the equation

$$(3.15) \quad \left(r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \right)^2 \bar{u} + \frac{\partial^2}{\partial t^2} \bar{u} = \frac{P}{I} \delta[r - r_0(t)] / (r\alpha)$$

The boundary conditions at $r = 0$ and $r = R$ are eqs. (3.6), (3.7) and (3.8) with u replaced by \bar{u} . If the initial condition for u is $u = u_t = 0$ at $t = 0$, then the same conditions hold for \bar{u} , i.e.

$$(3.16) \quad \bar{u}(r, 0) = \bar{u}_t(r, 0) = 0$$

It should be noted that eq. (3.15) does not involve $\theta_0(t)$. Hence, the mean deflection $\bar{u}(r, t)$ is independent of the circumferential movement of the load and remains the same if the load P moving along any radial line with instantaneous position $r_0(t)$. $\bar{u}(r, t)$ is equal to the symmetric deflection of a circular plate with built-in edge at $r = R$, with same initial conditions and subjected to a load of total strength $P(2\pi/\alpha)$ uniformly distributed along the circle $r = r_0(t)$. The solution for the symmetric deflection can be constructed by making use of the eigenfunctions for the symmetric modes given in [12]. Procedures for the construction of the symmetric unsteady solution are described in [13] and [14].

4. The Unsteady Diffusion Equation

If $u(x,y,z,t)$ represents the temperature variation in a stationary medium or the mass fraction of species in a diffusion problem, u obeys the simple diffusion equation

$$(4.1) \quad Lu = \Delta u - C^{-2}u_t = f(x,y,z,t) \text{ in } D \text{ for } t > 0.$$

where C^2 the diffusivity constant. In addition to the boundary conditions of eqs.(2.3) and (2.11), there is one initial condition,

$$(4.2) \quad u(x,y,z,0) = q_0(x,y,z) \text{ in } D$$

Two examples are presented to illustrate how to select the curvilinear coordinates for a given domain and how to apply the Theorems and corollaries. These suggestions, which rely on the geometry of the domain, are also applicable to other problems of mathematical physics. Of course, the final explicit formulas will be different for different governing equations. The first example illustrates the use of Theorem II to obtain explicit solutions along the edge of a wedge, and those for the planar, the cylindrical and the spherical integrals in the wedge. The second example shows the use of Theorem II for the construction of the spherical mean in a domain bounded by two cones with a common vertex. The third example shows how to apply Theorem I to study the diffusion of pollutants in a moving medium and in particular to obtain directly the total flux of pollutants going upstream.

Example 1. Three dimensional heat transfer problem in a wedge

Figure 3 shows that the domain D is a wedge with angle

μ . The edge is chosen as the z -axis. One face of the edge is the x - z plane ($\phi=0$). Due to the method of superposition and the Duhamel's principal, it suffices to consider the simple case of homogeneous equation and boundary data while the initial data is a point source located at a point $P(x_1, y_1, 0)$ with strength Q , i.e., $f = 0$, $g = 0$ and

$$(4.3) \quad q_0(x, y, z) = Q \delta(x-x_1, y-y_1, z)$$

with $x_1 = \rho_1 \cos \phi_1$, $y_1 = \rho_1 \sin \phi_1$, $\rho_1 > 0$ and $0 < \phi_1 < \mu$.

In absence of the wedge, the solution u^* of the homogeneous heat conduction equation in the whole space under the initial condition of eq.(4.3) is [15]

$$(4.4) \quad u^*(x, y, z, t) = Q(4\pi C^2 t)^{-3/2} \exp\{-r_1^2/(4C^2 t)\}$$

where r_1 is the distance from (x, y, z) to the location of the source.

In term of cylindrical coordinates ρ, ϕ , and z , z can be identified as the ξ coordinate and the planar integral of the solution u in the wedges at constant z is equal to that of u^* in the whole space by Corollary 1, i.e.

$$\begin{aligned} I_P(z, t) &= \int_0^\infty \rho d\rho \int_0^\mu d\phi \, u(\rho, \phi, z, t) \\ &= \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \, u^*(x, y, z, t) \\ (4.5) \quad &= Q(4\pi C^2 t)^{-1/2} \exp[-z^2/(4C^2 t)] \end{aligned}$$

If now ρ is identified as ξ . A constant ξ -surface is

a cylindrical surface with radius ρ . Corollary 1 gives immediately the cylindrical surface integral of u , i.e., [7]

$$\begin{aligned}
 I_c(\rho, t) &= \int_{-\infty}^{\infty} dz \int_0^{\mu} \rho d\phi u(\rho, \phi, z, t) \\
 &= \int_{-\infty}^{\infty} dz \int_0^{2\pi} \rho d\phi u^*(\rho \cos \phi, \rho \sin \phi, z, t) \\
 &= Q \rho (2 C^2 t)^{-1} I_o[\rho \rho_1 / (2 C^2 t)] \exp\{-(\rho^2 \\
 (4.6) \quad &+ \rho_1^2) / 4 C^2 t\}
 \end{aligned}$$

Since any point $A(0, 0, z_o)$ on the edge can be identified as the vertex of a cone with solid angle 2μ the point A can be chosen as the origin of the spherical coordinates (r, θ, ϕ) . The value of u at A is therefore related to u^* by corollary 2,

$$\begin{aligned}
 u(0, 0, z_o, t) &= \frac{4\pi}{2\mu} u^*(0, 0, z_o, t) \\
 (4.7) \quad &= (2\pi/\mu) Q(4\pi C^2 t)^{-3/2} \exp\{-[\rho_1^2 + (z_1 - z_o)^2] / (4 C^2 t)\}
 \end{aligned}$$

The spherical integral of u with A as the center is also related to that of u^* as follows

$$I_s(r, z_o, t) = r^2 \int_0^{\pi} \sin \theta d\theta \int_0^{\mu} d\phi u = r^2 \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi u^*$$

or

$$\begin{aligned}
 I_s(r, z_o, t) &= Q(4\pi C^2 t)^{-1/2} (r/r_p) \{ \exp[-(\eta - \eta_1)^2] \\
 (4.8) \quad &- \exp[-(\eta + \eta_1)^2] \}
 \end{aligned}$$

where $\eta^2 = r^2 / (4 C^2 t)$, $\eta_1^2 = r_p^2 / (4 C^2 t)$

and $r_p = [\rho_1^2 + (z_1 - z_0^2)]^{\frac{1}{2}}$

is the distance from the point source P to A.

Explicit formulas are obtained for the solution along the edge in eq.(4.7) and for the integrals of the solution on a constant z-plane, and a cylindrical surface with z-axis as its axis and with radius ρ and on a spherical surface of radius r centered at a point $(0,0,z_0)$ on the edge in eqs.(4.5),(4.6) and (4.8) respectively. The only missing information about the solution is its dependence on the variable ϕ . The success of obtaining so many explicit results from the corollaries is due to the special geometry of the domain, a wedge, and is, therefore, not restricted to diffusion problem only.

Example 2. Temperature field in a solid bounded by two insulated conical surfaces with a common vertex.

As shown in fig.4, the inner and the outer conical surfaces are nonintersecting. Their solid angles are Ω_1 and Ω_0 respectively with $\Omega_0 > \Omega_1$. With the boundary condition of $\partial u / \partial n = 0$ on the two conical surfaces, theorem II and the corollaries remain applicable with the solid angle of the domain D equal to $\Omega_0 - \Omega_1$. Based on the reason stated in the first example, it suffices to consider only the simple case of a point source of strength Q at a point P in D at the instant $t = 0$.

The solution u^* in the whole space without the conical boundaries is again given by eq.(4.4). The temperature u at the common vertex A is again given by the corollary 2.

$$\begin{aligned}
 u(A,t) &= \frac{4\pi}{\Omega_0 - \Omega_1} u^*(A,t) \\
 (4.9) \quad &= \left(\frac{4\pi}{\Omega_0 - \Omega_1} \right) \frac{Q}{(4\pi C^2 t)^{3/2}} e^{-\frac{r_1^2}{4C^2 t}}
 \end{aligned}$$

where $r_1 = |PA|$ is the distance from the point source to the common vertex A. The integral of the temperature over the spherical surface S_r in D with radius r and centered at A is again given by the expression on the right side of eq.(4.8).

When the conical surfaces are of finite length ξ_2 the domain D is bounded in addition by an insulated spherical surface S_2 with radius ξ_2 . The associated solution u^* should be replaced by the solution in an insulated sphere of radius ξ_2 due to a point source at P at the instant $t = 0$. The value at the vertex $u(A,t)$ and the spherical mean can be related by Theorem I to a spherically symmetric problem given in [15].

The axially symmetric temperature field between two co-axial circular cones is investigated by Oberhettinger and Dressler [16]. Series solution is constructed for the isothermal boundary condition; $u = 0$. For case of adiabatic boundary condition, $\partial u / \partial n = 0$, only a brief outline of the method is presented. The explicit results in this example would be useful to check the numerical results of their analysis.

Example 3. Diffusion of pollutants in a moving medium.

To be more specific, the conical domain D will be an open channel connecting two large reservoirs. The channel is assumed to be of constant depth, h. The domain D is defined by

$$(4.10) \quad \xi_1 < \xi < \xi_2, -\alpha/2 < \eta < \alpha/2 \quad \text{and} \quad -h < \zeta = z < 0$$

For a constant area channel $\xi = x$ and $\eta = y$; α is the width of the channel. For a divergence channel, $\xi = r$, and $\eta = \theta$; α is the divergent angle of the channel.

The flow is assumed to be incompressible and the velocity field can be represented as

$$(4.11) \quad \vec{q} = \xi V (\xi_1/\xi)^\lambda$$

where V is the velocity at the section $\xi = \xi_1$; $\lambda = 0$ for a constant area channel and $\lambda = 1$ for a divergence channel.

The governing equation for the diffusion of a pollutant in the channel is [15].

$$(4.12) \quad \frac{\partial u}{\partial t} + V \left(\frac{\xi_1}{\xi} \right)^\lambda \frac{\partial u}{\partial \xi} - C^2 \Delta u = C^2 f.$$

The boundary condition on S_c i.e., on the wetted walls of the channel and on the free surface is

$$(4.13) \quad \frac{\partial u}{\partial n} = 0.$$

The boundary conditions at the upstream and downstream ends of the channel will be

$$u = 0 \quad \text{at} \quad \xi = \xi_1 \quad \text{and} \quad \xi = \xi_2$$

because of the large sizes of reservoirs. The inhomogeneous term f represents the source of pollutant in the channel. For unsteady problems an initial condition, say $u = 0$ at $t = 0$, shall be added.

If the pollutant is dumped at a point ξ_0, η_0, z_0 in D at a rate $Q(t)$, then f can be written as

$$(4.14) \quad f = Q \delta(\xi - \xi_0) \delta(\eta - \eta_0) \delta(z - z_0) / \xi^\lambda$$

The mean solution $\bar{u}(\xi, t)$ obeys the equation

$$(4.15) \quad \frac{\partial \bar{u}}{\partial t} + v \left(\frac{\xi_1}{\xi} \right)^\lambda \frac{\partial \bar{u}}{\partial \xi} - c^2 \left[\frac{\partial^2 \bar{u}}{\partial \xi^2} + \xi^{-\lambda} \frac{\partial \bar{u}}{\partial \xi} \right] = c^2 Q \delta(\xi - \xi_0) / (\xi^\lambda \alpha)$$

The boundary conditions are

$$(4.16) \quad \bar{u} = 0 \text{ at } \xi = \xi_1 \text{ and } \xi_2$$

The mean solution $\bar{u}(\xi, t)$ can be obtained directly and is independent of η_0 and z_0 of the point source. From $\bar{u}(\xi, t)$, \bar{u}_ξ is obtained immediately and is the quantity of interest in this problem. In particular, \bar{u}_ξ at ξ_1 , and ξ_2 represent the rate of the pollutant entering the upstream and downstream reservoirs.

Explicit results will now be given for the steady problem to show the difference between a constant area channel and a divergent channel.

The steady solution of eqs. (4.15, 4.16) for a constant area channel ($\lambda = 0$) is

$$\bar{u}(\xi) = \begin{cases} \frac{Q}{\alpha V} \frac{e^{-(\xi_0 - \xi)k} - e^{-(\xi_0 - \xi_1)k}}{[1 - e^{-(\xi_2 - \xi_0)k}]} \frac{1}{[1 - e^{-kX}]} & \text{for } \xi_0 < \xi < \xi_1 \\ \frac{Q}{\alpha V} \frac{e^{-(\xi_2 - \xi)k}}{[1 - e^{-(\xi_0 - \xi_1)k}]} \frac{1}{[1 - e^{-kX}]} & \text{for } \xi_2 < \xi < \xi_0 \end{cases}$$

where $k = V/C^2$ and $X = \xi_2 - \xi_1$. The percentage of the pollutant reaching the upstream reservoir is

$$(4.17) \quad \frac{\alpha C^2}{Q} \bar{u}'(\xi_1) = \frac{e^{-(\xi_0 - \xi_1)k} [1 - e^{-(\xi_2 - \xi_0)k}]}{[1 - e^{-kX}]} \sim e^{-(\xi_0 - \xi_1)k} \quad \text{as } k \rightarrow \infty$$

The percentage decreases exponentially in the distance from the source of the pollution to the upstream station of the channel.

For a divergent channel ($\lambda = 1$, $V > 0$), the steady solution is

$$(4.18) \quad \bar{u}(\xi) = \frac{Q}{\alpha \xi_0 \text{Re}} \left\{ \left[\left(\frac{\xi}{\xi_1} \right)^{\text{Re}} - 1 \right] \left[\left(\frac{\xi_2}{\xi_0} \right)^{\text{Re}} - 1 \right] / \left[\left(\frac{\xi_2}{\xi_1} \right)^{\text{Re}} - 1 \right] - \frac{\xi_0}{\xi_1} \left[\left(\frac{\xi}{\xi_0} \right)^{\text{Re}} - 1 \right] H(\xi - \xi_0) \right\}$$

where $\text{Re} = V\xi_1/C^2$. The percentage of the pollutant going to the upstream reservoir is

$$(4.19) \quad \frac{\bar{u}'(\xi_1) \xi_1 \alpha}{Q} = \frac{\left(\frac{\xi_2}{\xi_0} \right)^{\text{Re}} - 1}{\left(\frac{\xi_2}{\xi_1} \right)^{\text{Re}} - 1} = \left(\frac{\xi_1}{\xi_0} \right)^{\text{Re}} \left[\frac{1 - \left(\frac{\xi_0}{\xi_2} \right)^{\text{Re}}}{1 - \left(\frac{\xi_1}{\xi_2} \right)^{\text{Re}}} \right]$$

As $\text{Re} \rightarrow \infty$, the percentage decreases as the Re -th power of the ratio ξ_1/ξ_0 .

When $V < 0$, the flow is reversed from $\xi = \xi_2$ to $\xi = \xi_1$ through a convergent channels. Equations (4.18) and (4.19) remain valid for $V < 0$, however, eq. (4.19) defines now the percentage of pollutant going to the

downstream reservoirs and

$$(4.20) \quad \frac{\bar{u}^*(\xi_1) \xi_1}{Q} \alpha \sim 1 - \left(\frac{\xi_0}{\xi_2}\right)^{-Re} \quad \text{as } -Re \rightarrow \infty$$

The percentage going to the upstream reservoir can be computed from $\xi_2 \propto \bar{u}^*(\xi_2)/Q$ or can be seen from (4.20). It again decreases as the $(-Re)$ -th power of the ratio (ξ_0/ξ_2) as $-Re \rightarrow \infty$ in contrast with the exponential decay law in a constant area channel.

5. The Wave Equation

The propagation of acoustic waves or that of a single component of electro or magnetic waves is governed by the simple wave equation

$$(5.1) \quad Lu = \Delta u - C^{-2} u_{tt} = f(x, y, z, t)$$

where C is the speed of propagation. The usual two initial conditions are the prescription of u and u_t at a given instant say $t = 0$.

In diffraction problems, it is usually done to define an incident wave $u^{(i)}(x, y, z, t)$ and to designate $t = 0$ as the instant the incident wave front hits the body. In other-words, the support of $u^{(i)}$ does not intersect the body for $t \leq 0$. The solution $u(x, y, z, t)$ for $t > 0$ D, i.e., in the domain exterior to the body is a solution of the wave equation, eq. (5.1) subjected to the initial conditions of

$$(5.2) \quad u(x, y, z, 0) = u^{(i)}(x, y, z, 0)$$

and

$$(5.3) \quad u_t(x, y, z, 0) = u_t^{(i)}(x, y, z, 0)$$

and the boundary condition

$$(5.4) \quad u_n = 0 \quad \text{on} \quad S_c$$

where S_c is the surface of the body. When the body is a cone of solid angle Ω , its exterior angle Ω_c , which is $4\pi - \Omega$, is the solid angle for the domain D . Since the incident wave $u^{(i)}$ is a solution the wave equation in the whole space without the cone

for $t > 0$ with the same initial conditions of eqs.(5.2) and (5.3), $u^{(i)}$ can be identified as the associate solution u^* in the whole space D^* and corollaries 1 and 2 yield

$$(5.5) \quad \iint_{S_r} u ds = \iint_{S_r^*} u^{(i)} ds$$

and

$$(5.6) \quad u(o,o,o,t) = \frac{4\pi}{4\pi - \Omega} u^{(i)}(o,o,o,t)$$

with S_r^* in D^* . Equation (5.6) is the result obtained previously [1] by a different method.

If the resultant wave is written as the sum of the incident wave and a secondary wave, which is the reflected and the diffracted waves, i.e.,

$$(5.7) \quad u = u^{(i)} + w$$

eq.(5.5) becomes

$$(5.8) \quad \iint_{S_r} w dS = \iint_{S_r^* - S_r} u^{(i)} ds$$

These equations can be summarized in the following statement.

Theorem III

When an incident wave is reflected and diffracted by a cone of solid angle Ω after the instant $t = 0$, the spherical integral of the incident wave cut off by the cone is redistributed exterior to the cone as the integral of the reflected and the diffracted waves. The redistribution of the incident wave from a spherical surface of angle 4π to that of $4\pi - \Omega$ intensifies the value at the vertex by the ratio $4\pi/(4\pi - \Omega)$.

If an instant (say $t = 0$) cannot be found such that the incident wave is neither reflected nor diffracted by the core, theorem III cannot be applied since the initial data cannot be stated in terms of the incident wave alone. In other words, Theorem II cannot be employed to relate the solution u in D , i.e., exterior to a conical body D_D , to $u^{(i)}$ in the whole space without the body. However, it still can be used in diffraction problems to relate the solution u to a known solution u^* in D^* which is exterior to another conical body D_D^* with the same vertex as stated in the following theorem:

Theorem IV

If the support of the solution u^* outside a conical body D_D^* does not intersect the conical body D_D for $t \leq 0$ and the surface of the cone D_D contains the part of the surface of D_D^* where the incident wave is reflected or diffracted for $t \leq 0$, then the solution u in D , fulfills the following relationships

$$(5.9) \quad u \equiv u^* \quad \text{for} \quad t \leq 0,$$

$$(5.10) \quad \iint_{S_r} u \, dS = \iint_{S_r^*} u^* \, dS, \quad \text{for all } t$$

and

$$(5.11) \quad u(0, t) = \left(\frac{4\pi - \Omega^*}{4\pi - \Omega} \right) u^*(0, t), \quad \text{for all } t$$

where S_r and S_r^* are the spherical surfaces outside the conical bodies D_D and D_D^* respectively. The spherical surfaces are centered at point O , the vertex, and the solid angles of the conical bodies are Ω and Ω^* respectively.

The two solutions u and u^* differ from each other for

$t > 0$ because that the reflection and diffraction will then take place over different surfaces. However, u and u^* have the same initial data at $t = 0$, therefore, Theorem II is applicable and eqs.(5.10) and (5.11) are valid for all t .

Theorems III and IV are special adaptations of Theorem II for diffraction problems. Their applications are demonstrated in the following three examples. The first example shows a direct application of Theorem IV to a trihedral and the explicit formulas become useful for the more complicated problems in the next two examples. The second and the third examples deal with bodies which are composed of cones of finite length. These examples illustrate how to apply the Theorems to each conical component for a finite interval due to the finite speed of propagation.

Example 1. Diffraction by a trihedral.

Figure 5 shows an incident wave which is diffracted all the time by one edge (OA) of a trihedral with solid angle Ω . The primary wave $u^{(p)}$ will pass over the vertex or one of the other two edges OB and OE at the instant $t = 0$. The given primary wave $u^{(p)}$ represents the incident and the diffraction waves advancing over a dihedral of angle χ formed by the two plane OAE and OAB. The solution u exterior to the trihedral will differ from $u^{(p)}$ only for $t > 0$.

If the planes OAE and OAB are extended beyond the edges OB and OE to form a wedge in place of the trihedral, $u^{(p)}$ will be the solution also for $t > 0$. Point O which lies on the edge of the wedge can be considered as the vertex of a conical body of solid angle 2χ . With u^* identified as $u^{(p)}$ and Ω^* as 2χ , Theorem IV yields

$$(5.12) \quad u(0,t) = \frac{4\pi - 2\chi}{4\pi - \Omega} u^{(p)}(0,t)$$

and

$$(5.13) \quad \iint_{S_r} u \, dS = r^2 \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi-\chi} d\phi u^{(p)}(r,\theta,\phi,t)$$

where r is the distance to the vertex 0, S_r is the spherical surface outside the trihedral, $\theta = 0$ is the edge OA and $\phi = 0$ is the plane OAB.

Example 2. Diffraction of a plane pulse by a rectangular block
(a two dimensional problem)

As shows in Figure 6, a plane pulse of strength ϵ hits the first corner 0 of the block at the instant $t = 0$ at an angle λ with respect to the horizontal side OA. The pulse is reflected by the two sides of the corner and the influence of the vertex is confined by the circle of radius Ct . The solution outside the circle and behind the reflected wave is 2ϵ . The solution inside the circle is a conical solution [17] in two variables, $u_c(\bar{x}, \bar{y})$, with $\bar{x} = x/(Ct)$ and $\bar{y} = y/(Ct)$. Before the construction of the conical solution, the "spherical" integral and the value at the vertex 0 will be related directly to the incident wave.

For the two dimensional problem, cylindrical coordinates, ρ, θ with the vertex 0 as the origin will be used. The "spherical" integral will become circular integral of radius ρ . Theorem III, that is, eqs. (5.5 and 5.6) yields the circular integral of the solution u due to the diffraction by the first corner at 0

$$(5.14) \quad \begin{aligned} \int_0^{3\pi/2} u_1(\rho, \theta, t) \rho d\theta &= \int_0^{2\pi} u^{(i)}(\rho, \theta, t) \rho d\theta \\ &= \begin{cases} \epsilon(2\pi\rho) & \text{for } \rho \leq Ct \\ \epsilon[2\pi - 2 \cos^{-1}(Ct/\rho)]\rho & \text{for } \rho > Ct > 0 \end{cases} \end{aligned}$$

and the value at the vertex 0,

$$(5.15) \quad u_1(0, t) = \frac{2\pi\epsilon}{3\pi/2} = \frac{4}{3} \epsilon \quad t > 0$$

Equation (5.14) is valid only when the circle of radius ρ does not receive the diffracted waves from the adjacent corners.

This condition creates an upper time limit for the validity of eq.(5.14). It is

$$(5.16) \quad Ct \leq L(1 + \sin \lambda) - \rho$$

For equation (5.15), the time limit is given by eq.(5.16) with $\rho = 0$. For the convenience of explanation, the width L of the block is assumed to be shorter than its height H and the incident pulse arrives at the corner $A(L, 0)$ before at the corner $B(0, H)$ i.e., $L \sin \lambda < H \cos \lambda$ is also assumed.

If the conical solution $u_c(\bar{x}, \bar{y})$ is constructed by the method of [17], it defines the solution u_1 for the domain exterior to the first corner at 0. Theorem IV can be used to relate the diffracted wave at the adjacent corner A, to u_1 when $t > L \sin \lambda / C$. For a finite duration, corner A can be considered as the vertex of a wedge. In terms of the cylindrical coordinates ρ_1 and θ_1 , the domain D exterior to corner A is $-\pi/2 < \theta_1 < \pi$ and $\rho_1 > 0$. The associated domain will be the space above the line A0, and its extension through A as shown by the dotted line in Fig.6, i.e. $0 < \theta_1 < \pi$ and $\rho_1 > 0$. For the associated domain D*, there is no diffraction at the corner A, therefore, the associated solution can be identified as u_1 for $t > L \sin \lambda / C$. The circular integral of the solution u_2 exterior to the corner A is given by that of u_1 ,

$$\int_{-\pi/2}^{\pi} u_2(\rho_1, \theta_1, t) d\theta_1 = \int_0^{\pi} u_1(\rho_1, \theta, t) d\theta_1$$

for $2L + L \sin \lambda - \rho_1 \geq Ct > L \sin \lambda$. The ratio of the angles of the domain D^* and D at the vertex A is $\pi/(3\pi/2) = 2/3$. The value at the vertex A is

$$u_2(A, t) = \begin{cases} \frac{2}{3} (2\varepsilon) = \frac{4}{3} \varepsilon & \text{for } L \geq Ct > L \sin \lambda \\ \frac{2}{3} u_c\left(\frac{L}{Ct}, 0\right) & \text{for } 2L + L \sin \lambda \geq Ct \geq L \end{cases}$$

These equations can also be obtained by applying the results of example 1. It is clear that if the solution u_2 after the diffraction at corner B is constructed by the method in [18] Theorem IV will be useful for the subsequent diffraction of u_2 at an adjacent corner.

Example 3. Diffraction of a plane pulse by a cube.

Figure 7 shows a plane pulse of strength ε hitting the first corner O of a cube at the instant $t = 0$. The edges OA , OB and OE are chosen as the x_1 -, x_2 -, and x_3 - axes respectively. The directional cosines of the normal to the plane pulse are n_1 , n_2 and n_3 . They are all non-negative and the smallest one is n_1 . Before the arrival of the plane pulse at any other corner, i.e., $t < n_1 L/C$, the solution $u_1(x_i, t)$ will remain unchanged if the cube is replaced by the three dimensional corner at O with solid angle $\Omega = 4\pi/8 = \pi/2$. Equations (5.5) and (5.6) yield the spherical integral of u_1 outside the corner,

$$\begin{aligned} \iint_{S_r} u_1(x_i, t) ds &= \int_0^{\pi} d\theta \int_{-\pi}^{\pi} u^{(i)} r^2 \sin \theta d\phi \\ &= \begin{cases} \varepsilon r^2 4\pi & \text{for } r \leq Ct \\ \varepsilon r[r + Ct] 2\pi & \text{for } r > Ct > 0 \end{cases} \end{aligned}$$

and for $Ct < (1 + n_1)L - r$ and the value at the vertex 0

$$u_1(0,t) = \frac{4\pi\epsilon}{4-\pi/2} = \frac{8}{7}\epsilon \quad \text{for } (1 + n_1)L > Ct > 0$$

The upper time limits are defined by the arrival of the diffracted waves from an adjacent corner. Along the edges and outside the domain of influence of the vertex i.e., $r > Ct$, the value is same as that along the edge of a 90° wedge, i.e.,

$$u_1(x_i, 0, 0, t) = (4/3)\epsilon \quad \text{for } Ct/n_i > x_i > Ct,$$

$$i = 1, 2, 3.$$

To obtain the complete solution $u_1(x, y, z, t)$ it is necessary to apply the method of [17] for conical solutions of two variables outside the characteristic sphere $r = Ct$ and the method of [4] for the three dimension conical solutions inside the sphere. After the determination of the solution u_1 for the diffraction by the corner 0, Theorem IV and in particular, the results in example 1 can be applied to the solution of diffraction of u_1 by the adjacent corners.

When $t > n_1 L/C$, the wave front of u_1 advances over the edges AB' and AE' and the vertex A. A can be considered as the vertex of a trihedral with solid angle $\Omega = \pi/2$, and OA as the edge of a dihedral with angle $\chi = \pi/2$. The results of example 1 i.e., eqs. (5.12) and (5.13) are applicable until the arrival of diffracted waves from other corners. The results are

$$(5.17) \quad u_2(A, t) = 6/7 \quad u_1(A, t)$$

and

$$(5.18) \quad \iint_{S_{r'}} (u_2 - u_1) dS = 0$$

where u_2 is the solution after the diffraction of u_1 by the corner at A, $S_{r'}$ is the spherical surface outside the trihedral with radius r' to its vertex A and the extension of u_1 outside its domain of definition is equal to zero. Similarly, eqs. (5.12) and (5.13) can be applied for any point on the edge AB' (or AE') before the arrival of the diffracted waves from the vertex A or B'.

The applicability of eqs. (5.12) and (5.13) to vertex A to obtain eqs. (5.17) and (5.18) does not depend on the facts that u_1 is a conical solution with respect to 0 and that 0 and A are vertices of a cube. It is obvious that the results of example 1 can be applied to relate the solution u^* of diffraction of a wave by a vertex A of a polyhedron to the solution u of the subsequent diffraction of u^* by an adjacent vertex B of the polyhedron. If the polyhedron is convex, the solution u can be written as $u^* + u^{(d)}$. The extension of u^* outside its domain of definition equals to zero. There is a discontinuity in u^* and also in $u^{(d)}$ but their sum is continuous. Theorem IV or Eqs. (5.17 and 18) yield

$$(5.19) \quad \iint_{S_r} u^{(d)} ds = 0$$

and

$$(5.20) \quad \begin{aligned} u^{(d)}(B, t) &= u(B, t) - u^*(B, t) \\ &= - \frac{2\chi - \Omega}{4\pi - \Omega} u^*(B, t) \end{aligned}$$

where χ is the angle of the dihedral with edge AB and Ω is the solid angle of the polyhedron at the vertex B. Of course, eqs. (5.19 and 5.20) will remain valid only before the arrival of additional diffracted waves from the adjacent corners.

6. The Reduced Wave Equation

For periodic solutions of wave equation $v(x,y,z)e^{i\omega t}$, the amplitude function v is governed by the reduced wave equation

$$(6.1) \quad \Delta v + k^2 v = f \quad \text{in } D$$

where $k^2 = c^2 \omega^2$.

If the domain D is bounded and in particular a truncated cone ($\xi_2 < \infty$) the governing equation for the spherical mean $\bar{v}(r)$ is an ordinary differential

$$(6.2) \quad (\bar{v}' r^2)' + k^2 r^2 \bar{v} = \bar{f}$$

There is no difficulty to obtain the spherical mean $\bar{v}(r)$ for $\xi_1 \leq r \leq \xi_2$ and the value at the vertex of the cone when $\xi_1 = 0$.

When the domain D is unbounded, there is the well-known difficulty of what is the proper boundary condition at infinity. This difficulty does not appear for the solution of an initial boundary value problem of the wave equation.

For the diffraction of a incident periodic wave with amplitude $v^{(i)}$ by a obstacle, the solution v can be represented in the form

$$(6.3) \quad v = v^{(i)} + w$$

where w is the secondary wave, the diffracted and reflected waves. When the obstacle is finite, the boundary condition at infinity is the well known Sommerfeld radiation conditions [19]

$$(6.4) \quad w = o(r^{-1}) \text{ and } w_r + i k w = o(r^{-1})$$

The conditions can be deduced by treating the periodic solution

as limit of an aperiodic solution for example the principle of limiting absorption or limiting amplitude [20].

If the obstacle is unbounded, say a half plane, a wedge or a cone, and the incident wave $v^{(i)}$ itself fulfills eq.(6.4), e.g. $v^{(i)}$ is induced by a point source, or a source distribution of bounded support, the radiation conditions of eq.(6.4) for the secondary wave remain valid [21].

If the incident wave itself does not fulfill the Sommerfeld radiation conditions, for example, the incident waves are plane waves or cylindrical waves, the secondary wave due to an unbounded obstacle may not obey the Sommerfeld conditions. These difficulties are mentioned in [22] and suggestions are made to separate the part of w which fulfill the Sommerfeld condition from the second part, which does not. Explicit determinations of the second part are demonstrated [22] for a plane and a half plane and can be done for a wedge. For a conical body with arbitrary cross-section, the determination of the second part in w is not obvious.

In the first subsection, the incident wave is assumed to fulfill the Sommerfeld conditions and so does the secondary wave. The corollaries of Theorem II are applied to relate the spherical mean of the resultant solution to that of the incident wave and to show that their relationships are the same as those of unsteady solutions of the wave equation. These relationships for the unsteady solutions will now be assumed to be valid for the periodic solution including the case that the incident wave does not fulfill the Sommerfeld conditions and yield the necessary conditions for the solution at infinity in the second subsection.

A. The incident wave fulfills the Sommerfeld conditions

Since $v^{(i)}$ and w both fulfill the Sommerfeld conditions, so does the resultant solution v in the domain D outside a conical body of solid angle Ω . Without the body, i.e. in the whole space D^* , then there is no secondary wave and the solution v^* is equal to $v^{(i)}$ and fulfills the same homogeneous boundary condition on $r = \xi_2 \rightarrow \infty$, i.e. eq.(6.4). Corollaries I and II are therefore applicable, and the results are:

$$(6.5) \quad \iint_{S_r} v \, dS = \iint_{S_r^*} v^{(i)} \, dS$$

and

$$(6.6) \quad v(0,0,0) = \frac{4\pi}{4\pi-\Omega} v^{(i)}(0,0,0)$$

where S_r^* is the spherical surface with radius r from the origin located at the vertex of the conical body and S_r is the part of the surface outside of the body. Equations (6.5) and (6.6) are identical to the relationships for eq.(5.5) and (5.6). In terms of the secondary waves w , eq.(6.5) becomes

$$(6.7) \quad \iint_{S_r} w \, dS = \iint_{S_r^* - S_r} v^{(i)} \, dS$$

This is identical to eq.(5.8) and has the same physical meaning stated in Theorem IV, i.e., the spherical integral of the secondary wave outside the body recovers the part of the spherical integral of the incident wave cut-off by the body.

B. The incident wave does not fulfill the Sommerfeld conditions

In this case, the proper boundary condition at infinity for the solution v or the secondary wave w is unknown, therefore,

the corollaries cannot be applied. However, if the limiting principle is accepted also for the diffraction by an unbounded body then the periodic solution $ve^{i\omega t}$ can be considered as the limiting solution ($t \rightarrow \infty$) of the wave equation with an incident wave which is periodic behind a wave front advancing towards the cone at finite time (say $t=0$). Equation (5.8) or Theorem IV which holds for all time should be valid also for the limiting solution. By this argument, eq. (6.7) is reestablished,

$$(6.8) \quad \iint_{S_r} w \, dS = \iint_{S_r^* - S_r} v_r^{(i)} \, ds$$

and also

$$(6.9) \quad \iint_{S_r} w_r \, dS = \iint_{S_r^* - S_r} v_r^{(i)} \, ds.$$

They become now the necessary conditions for the secondary wave w for any r including $r \rightarrow \infty$.

By looking at the Green's formula for the reduced wave equation [2] the integral over the spherical surface of large radius R is

$$\left[\iint_{S_R} v \, dS \right] \frac{\partial}{\partial r} \left(\frac{e^{-kr}}{4\pi r} \right) \Big|_{r=R} - \left[\iint_{S_R} v_r \, ds \right] \frac{e^{-kR}}{4\pi R}$$

The terms inside the first square bracket is recognized as the total strength of doublets on S_R and that inside the second one is that of sources on S_R . Equations (6.8) and (6.9) can therefore be interpreted as the following necessary conditions:

The total strength of sources induced by the secondary wave on the spherical surface of large radius R outside the conical body recovers that induced by the incident wave on the part of the

spherical surface cut off by the cone. The statement holds also for doublets.

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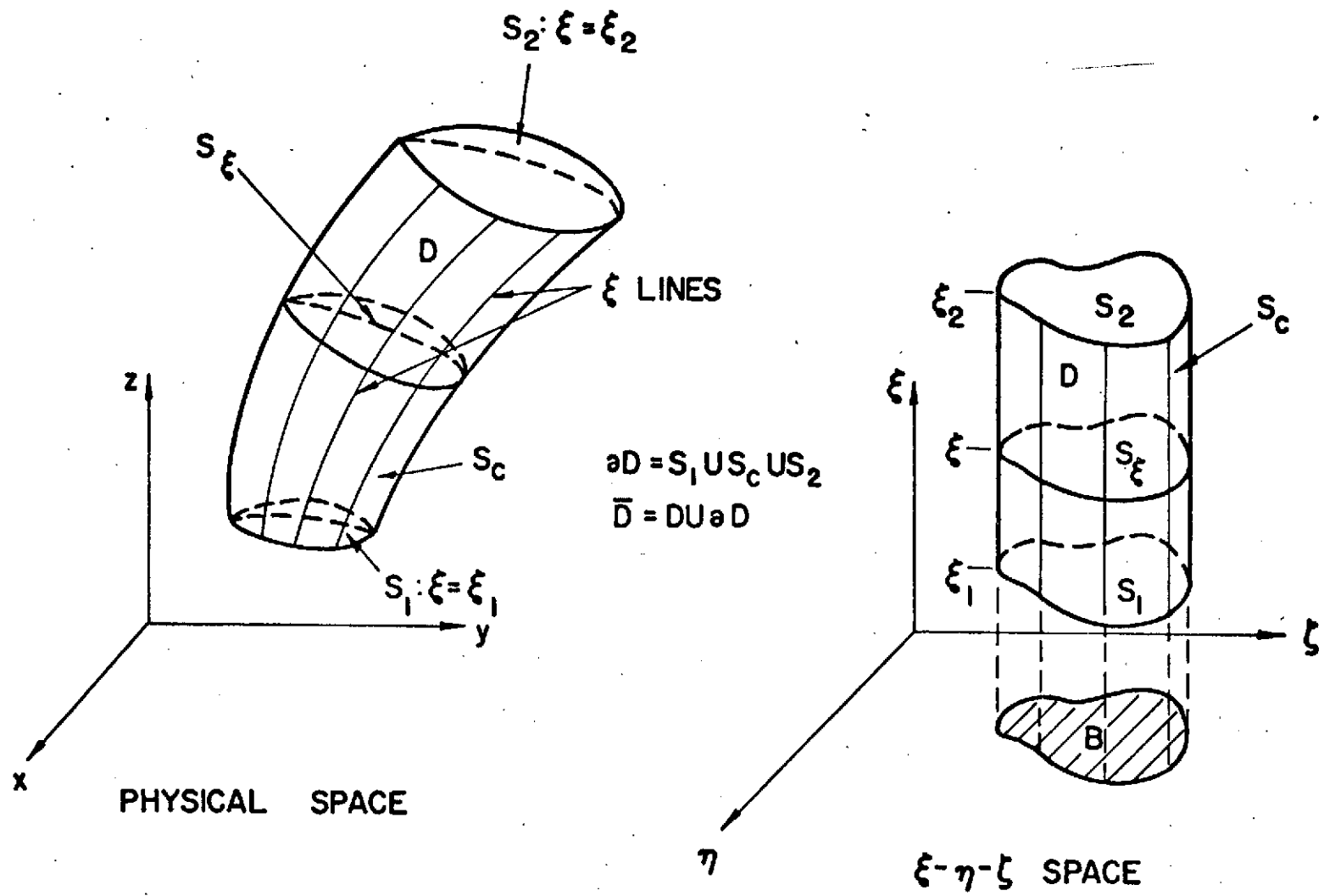


Fig. 1 The conical domain D and its boundary ∂D

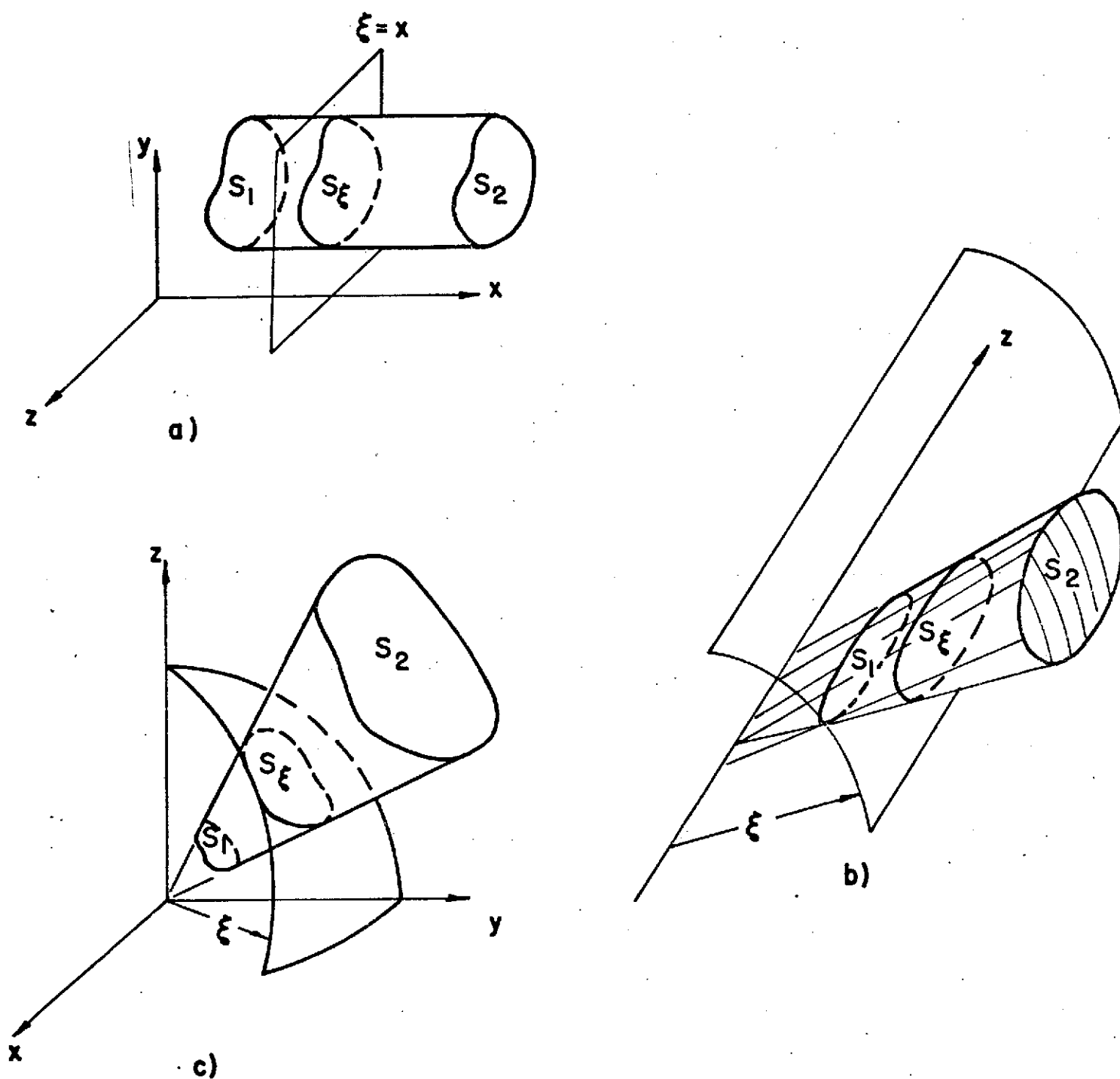


Fig. 2 Admissible Orthogonal Coordinates and the Constant ξ -Surfaces:
 (a) Parallel planes,
 (b) Co-axial circular cylindrical surfaces and,
 (c) Concentric spherical surfaces

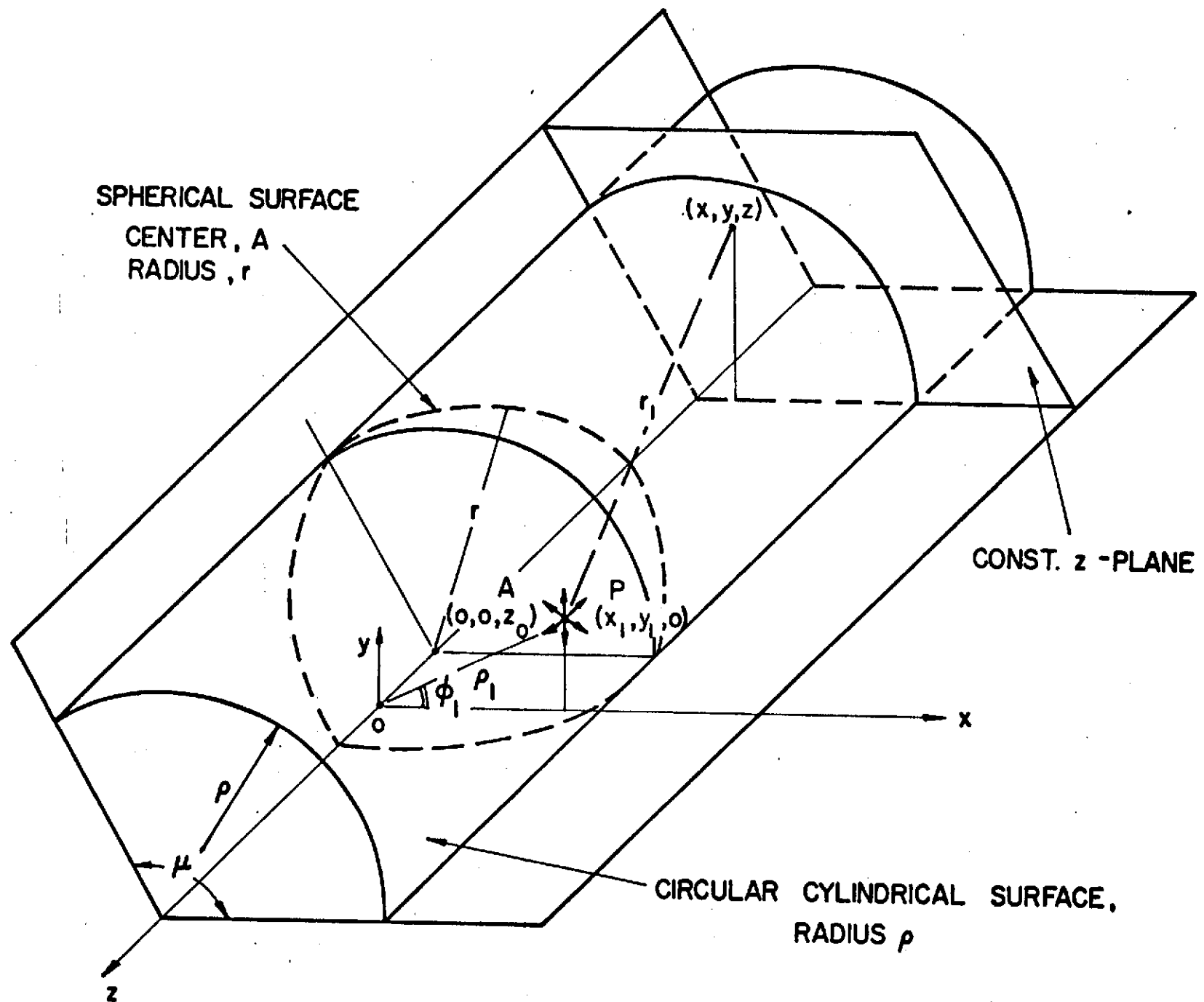


Fig. 3 Three Dimensional Problem in a Wedge

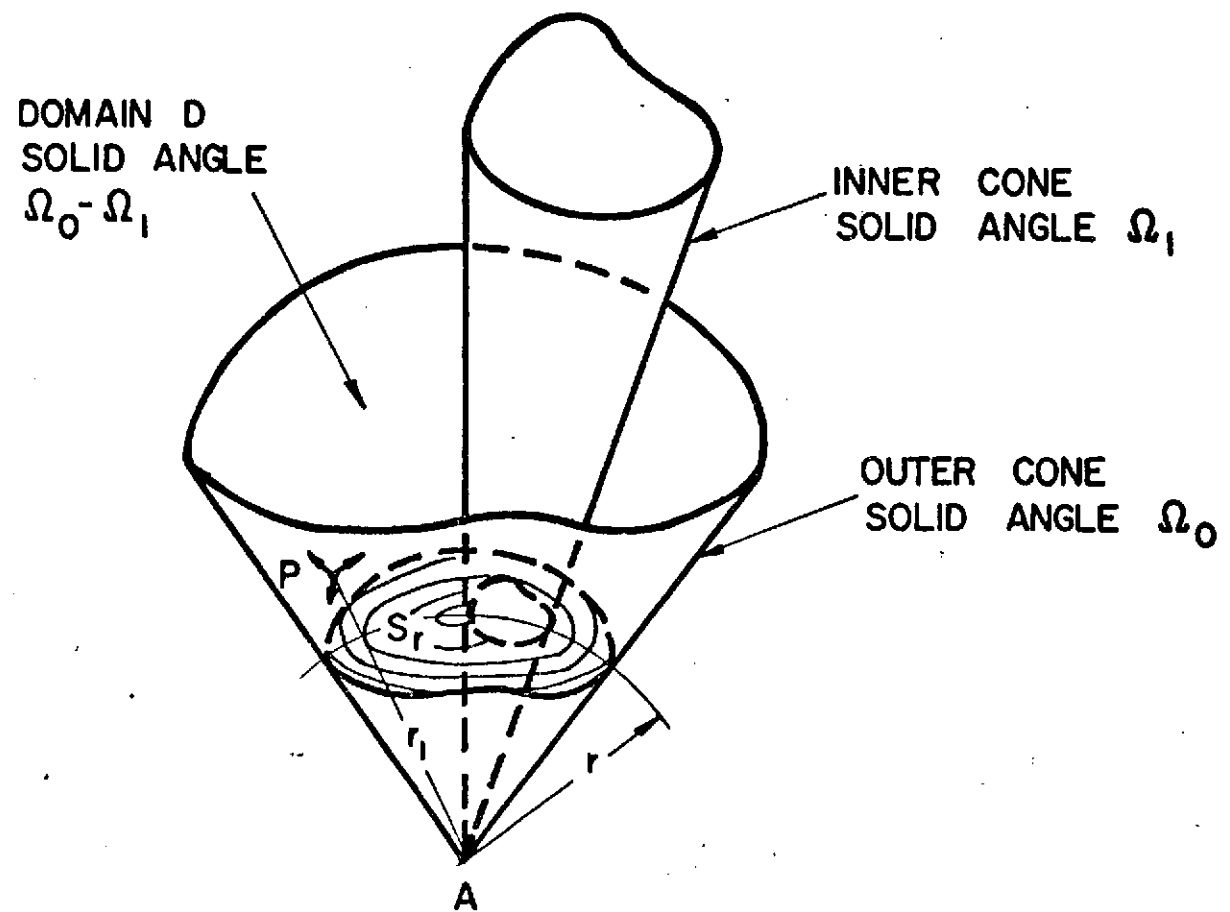


Fig. 4 Domain D bounded by two cones with a common vertex

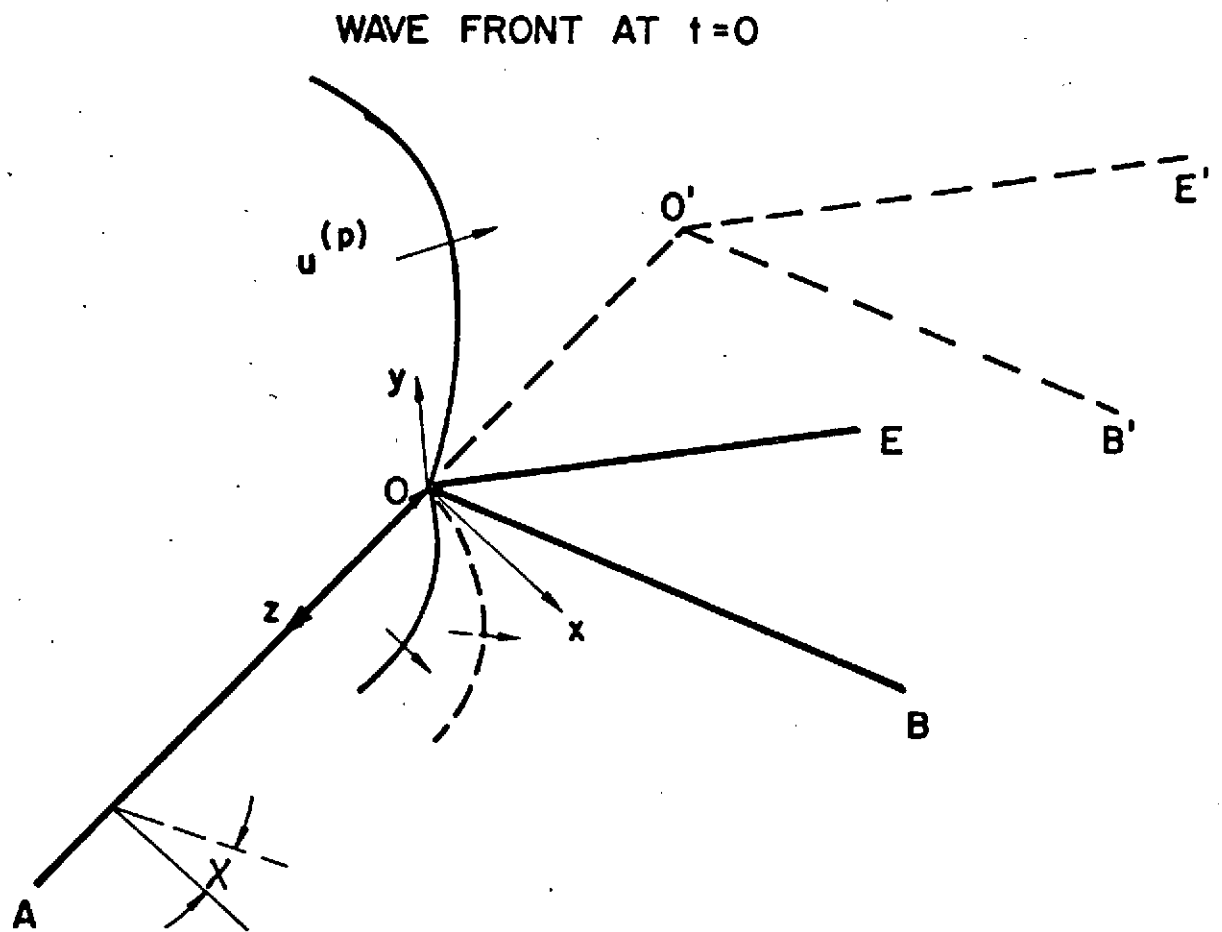
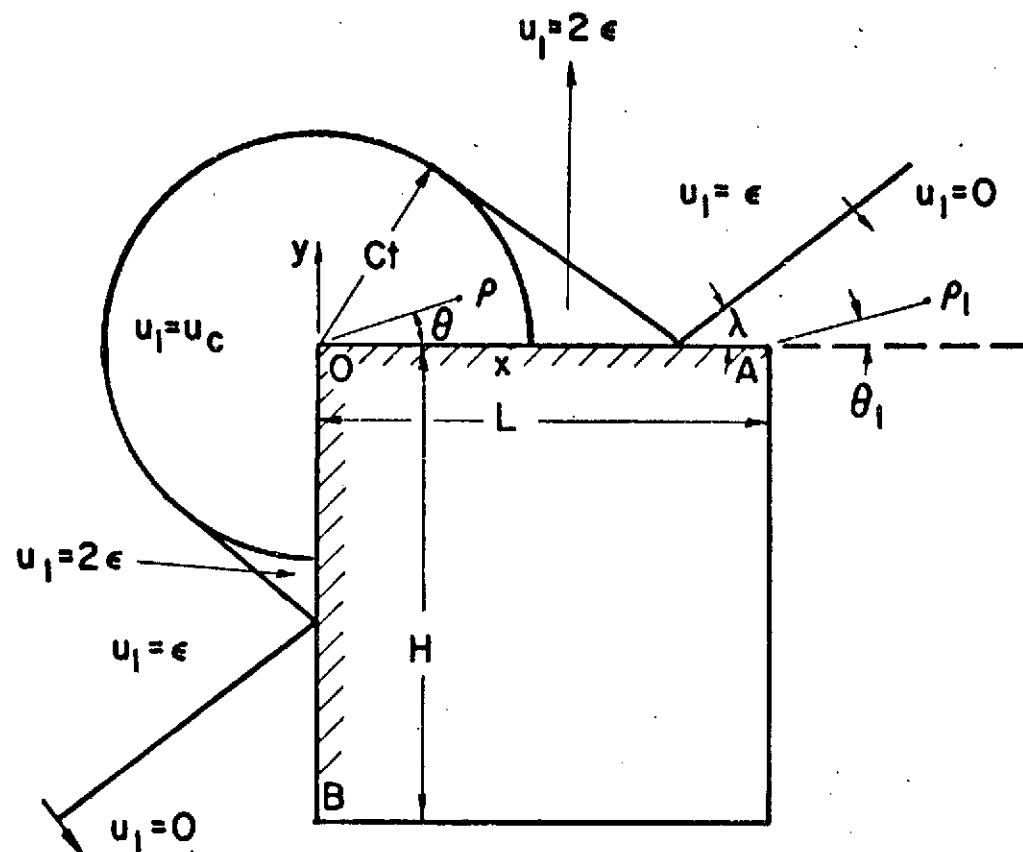


Fig. 5 Diffraction of a primary wave u^* by a trihedral $O ABE$ of solid angle Ω (Dotted lines show the extension of faces OAB and OAC to form a wedge of angle X)



• Fig. 6 Diffraction by a Rectangular Block (a two dimensional problem)

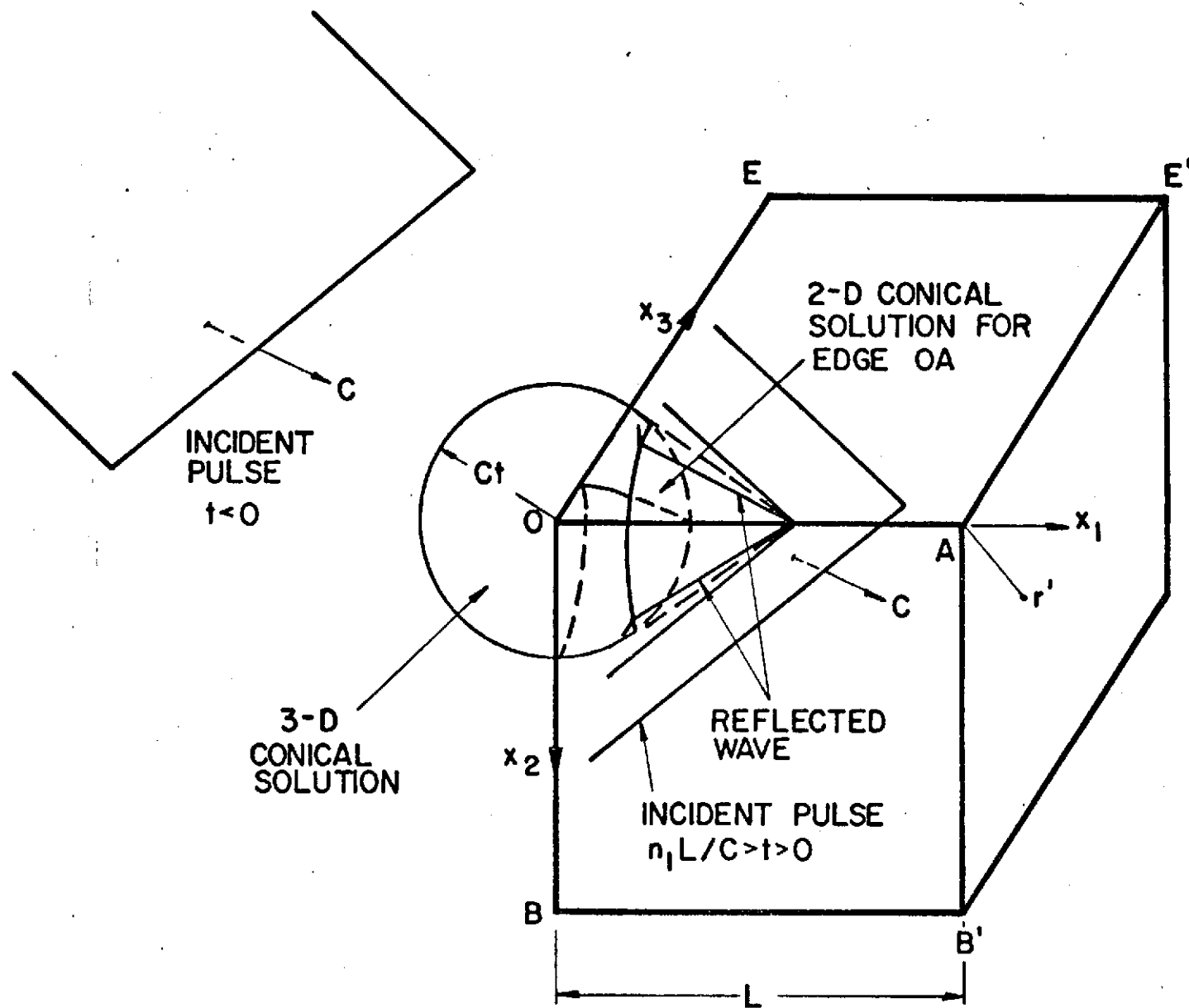


Fig. 7 Diffraction by a cube